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Frankl's Union-Closed Sets Conjecture

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***Abstract*: This study explores structural and pattern-based factors influencing the validity of Frankl's (Union-Closed Sets) Conjecture. By analyzing categorized families based on size, internal structure, and symmetry, we assess how these characteristics impact the conjecture's likelihood of holding. Particular focus is placed on families with high symmetry, dominant elements, or inherent union-closure. The results reveal consistent trends where certain configurations satisfy the conjecture trivially. These insights help pinpoint structural conditions that simplify the conjecture's verification and highlight families that may present greater complexity.**

***Keywords*: Frankl Conjecture, Union-closed sets conjecture, Union-closed family of sets.**

***Symbols:* F represents a union-closed family of sets.**

1. Introduction

The Frankl's Conjecture (also known as Union-closed sets Conjecture) is a problem posed by Pe´ter Frankl in 1979.

Frankl's Conjecture is rooted in the concept of union-closed families of sets: seeking to understand the structure of families of sets that are closed under unions. A family of sets F is said to be union-closed if, for any two sets $A, B \in F$, their union $A \cup B$ is also in F. The conjecture suggests that: *in any union-closed family of finite sets, there must be at least one element that appears in at least half of the sets.* This simple statement has led to a series of investigations into its validity. The first major steps in understanding the conjecture came from Frankl himself, who showed that the conjecture held for some small families and provided the first cases where it was not immediately clear whether the conjecture was true. Frankl's original proof was limited to specific families of sets, and he noted that more general methods would be needed to prove the conjecture for arbitrary families [13].

Some of the early breakthroughs was the proof for union-closed families where all sets are relatively small. For instance, when the family consists of only subsets of size one or two, the conjecture holds trivially. This was explored in works by Chva´tal (1982) [11], who showed that for small-sized families, it is easier to identify the element that appears in at least half of the sets. Similarly, research by Dukharev (1997) identified specific constructions where the conjecture is satisfied, providing examples for smaller families [19].

Proposition 1: Assume that F contains three different three-element sets which are all subsets of the same four-element set. Then F satisfies Frankl's conjecture.

Proposition 2: Suppose that F contains three three-element sets, each of which contains the same two elements. Then F satisfies Frankl's conjecture.

Proposition 3: Suppose that F contains three three-element sets, each of which contains the same two elements. Then F satisfies Frankl's conjecture.

Proposition 4: Let $\{a, b, c, d, e\} \subseteq X$, and let the sets $\{a, b, c\}$, $\{a, b, d\}$, $\{c, d, e\}$ belong to F. Then F satisfies Frankl's conjecture [16].

Theorem 1: Assume that F contains three different three-element sets which are all subsets of the same five-element set. Then F satisfies Frankl's conjecture [16].

Another special case is when the union of all sets in the family contains a limited number of distinct elements. For example, when the union of the sets has no more than 11 distinct elements, it has been proven that the conjecture holds [4]. This was first demonstrated in a computational approach by Brady (2002) who systematically checked families up to 11 elements using computational methods to show that in all such cases, there is an element that appears in at least half the sets.

Lemma 1: If $|X| = 11$ and F contains two three-element sets with a two-element intersection, then F satisfies Frankl's conjecture.

Lemma 2: If $|X| = 11$ and F contains three four-element subsets of a five-element set, then F satisfies Frankl's conjecture.

Lemma 3: Let $|X| = 11$ and F contain two four-element subsets of a five-element set. Then F satisfies Frankl's conjecture.

Lemma 4: Let $|X| = 11$ and F contain two three-element sets. Then F satisfies Frankl's conjecture.

Lemma 5: Let $|X| = 11$ and F contain a four-element set and one of its three-element subsets. Then F satisfies Frankl's conjecture.

Lemma 6: Let $|X| = 11$ and F contain a three-element set. Then F satisfies Frankl's conjecture.

Lemma 7: Let $|X| = 11$ and F contain a five-element set and one of its four-element subsets. Then F satisfies Frankl's conjecture.

Lemma 8: Let $|X| = 11$ and F contain a four-element set. Then F satisfies Frankl's conjecture.

Theorem 2: If $|X| = 11$, then F satisfies Frankl's conjecture [4].

In recent years, graph-theoretic and probabilistic methods have gained traction in studies related to Frankl's Conjecture. In particular, researchers have found that this conjecture can be

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analyzed through the lens of bipartite graphs, where one partition of the graph represents the sets in the family and the other represents the elements of the universal set.

In these bipartite graphs, an edge between a set and an element exists if the element is contained in the set. The union-closed property implies that certain graph-theoretic structures must exist, and these structures are helpful in demonstrating cases where Frankl's Conjecture holds. For instance, Bruhn and Schaudt (2015) applied graph-theoretic methods to show that for families of small sets, it is possible to find elements that appear in a sufficient number of sets by analyzing the independence number of the corresponding bipartite graphs [7], [6].

Gilmer (2022) [14] established the first constant lower bound for this conjecture,

Lemma 9: In any union-closed family of sets F, there exists an element that appears in at least $\frac{3-\sqrt{5}}{2} \approx 0.38$ of the sets in F.

In 2024, researchers Ryan Alweiss, Brice Huang, and Mark Sellke [1] made significant progress on Frankl's Conjecture by verifying an explicit inequality that was previously conjectured by Gilmer in 2022[14]. Their work provided a key improvement to the lower bound in the conjecture, demonstrating that for any nonempty union-closed family $F \subseteq 2^{[n]}$, there exists an element $i \in [n]$ that is contained in at least 38% of the sets in F.

Theorem 3: For all $\varphi \in [0, 1]$, the minimum of $F(\mu)$ over M_φ is attained at some μ supported on at most two points.

Furthermore, if a minimizer is supported on exactly two points, then one of the points is 0.

The case of μ supported on $\{0, x\}$ leads to the following definition:

$$S = \{\varphi \in [0, 1]: \varphi H(x^2) \geq xH(x) \ \forall x \in [\varphi, 1]\}, \ \varphi^* = \min(S).$$

Theorem 4: [1] The union-closed conjecture holds with constant $1 - \varphi^*$, i.e., for any non-empty union-closed family $F \subseteq 2^{[n]}$, some $i \in [n]$ is contained in at least $1 - \varphi^*$ fraction of the sets in F.

This result builds on previous advancements and further refines the understanding of the conjecture's validity. Specifically, it shows that at least one element in the family must appear in a significant portion of the sets—approximately 38%—regardless of the structure of the family. The improvement in this lower bound helps to solidify the conjecture and provides further insight into the minimum fraction of sets that must contain an element in any union-closed family.

In some cases, the conjecture is satisfied trivially due to structural properties like symmetry, dominance of certain elements, or inclusion of the full set. This study focuses on finding and understanding these types of patterns—especially those that make the conjecture hold "trivially." To do this, we examine families of sets created from universal sets with n-elements (that is, for $n = \{1, 2, \ldots, 6\}$). We analyze how aspects like the number of sets (size), the internal organization of the family (structure), and symmetry affect whether or not the conjecture holds. The goal is to identify consistent trends or features that influence the validity of Frankl's Conjecture.

2. Methods

To construct the possible subsets and identify union-closed families, we utilize Python scripts to automate the generation and analysis of subsets from the universal set $U = \{1, 2, 3, 4, 5, 6\}$. The full power set $P(U)$ comprising 64 subsets, is first generated. From this power set, candidate

union-closed families are formed by ensuring that for any two sets A and B within a family, their union $A \cup B$ is also included in the family.

The script further categorizes each union-closed family based on key structural attributes, including size (number of subsets), internal configuration, and symmetry. This categorization enables the rapid identification of structural or pattern-based trends that may influence the validity of Frankl's Conjecture, especially in larger families where manual analysis is impractical.

3. Patterns and Structures

A. F Contains Only One Non-Empty Set

Let $F = \{A\}$, where $A \subseteq U$ and $A \neq \emptyset$, be a union-closed family consisting of a single non-empty set.

Since there is only one set in F, every element $x \in A$ appears in all sets of the family. Therefore, each element in A appears in:

$\frac{1}{1} = 1$ (i.e., 100% of the sets).

This satisfies the Union-Closed Sets Conjecture, which requires that at least one element appears in at least half of the sets in the family.

Example:

Let $U = \{1, 2, \ldots, 6\}$

Consider a union-closed family F that includes only one non-empty set, say:

$$F = \{\{1, 2, 3\}\}$$

then the conjecture trivially holds because: Every element in the set appears in all of the sets in F. The conjecture requires that at least one element appears in at least half of the sets (which is just one set in this case). Since every element of $\{1\}$, $\{2\}$, $\{3\}$ appears in the only set present, the condition is satisfied.

The conjecture holds for all union-closed families containing exactly one non-empty set.

B. F Contains the Empty Set and One Non-Empty Set

Let F be a union-closed family over a universe U, such that:

$F = \{\emptyset, A\}$, *where* $A \subseteq U$ and $A \neq \emptyset$.

The total number of sets in F is 2. Each element $x \in A$ appears in exactly one of the two sets (namely, in A, but not in $\emptyset$). Thus, for every $x \in A$, the number of sets containing x is:

$$| \{B \in F \mid x \in B\} | = 1 = \frac{|F|}{2}.$$

Therefore, every element of A appears in exactly half of the sets in F, satisfying Frankl's Union-Closed Sets Conjecture.

The conjecture holds for all union-closed families consisting of the empty set and one non-empty set.

Example:

Let $U = \{1, 2, \ldots, 6\}$

Consider a union-closed family F that includes the empty set and at least one non-empty set, for example:

$$F = \{\emptyset, \{1, 2\}\}$$

The element 1 appears in only one set: {1, 2}. The element 2 also appears in only one set: {1, 2}. The total number of sets in F is 2. Each element of {1, 2} appears in at least $\frac{1}{2}$ of the sets (exactly one out of two). Thus, at least one element (actually both 1 and 2) appears in at least half of the sets, so the conjecture holds.

C. F Contains the Empty Set and a Singleton Set

Let F be a union-closed family of sets over a universe U, such that:

$F = \{\emptyset, \{x\}\}$

for some $x \in U$.

This family contains exactly two sets: the empty set and a singleton $\{x\}$. The element x appears in exactly one of the two sets. Therefore, the number of sets containing x is:

$$| \{A \in F \mid x \in A\} | = 1 = \frac{|F|}{2}$$

Hence, the element x appears in at least half of the sets in F, and Frankl's Union-Closed Sets Conjecture is satisfied. The conjecture holds for all union-closed families containing the empty set and a singleton set.

Example:

Let $U = \{1, 2, \ldots, 6\}$

Consider a union-closed family:

$F = \{\emptyset, \{3\}\}$

The total number of sets in F is 2. The element 3 appears in exactly one of these two sets. The fraction of sets containing element 3 is $\frac{1}{2}$, which satisfies the requirement meaning that the conjecture holds.

Thus, the conjecture holds for all union-closed families F containing a singleton set and the empty set.

D. F Contains the Full Power Set

Let $U = \{x_1, x_2, \ldots, x_n\}$ be a universal set with n elements. Then the power set P (U) contains 2^n subsets.

Theorem: For any fixed element $x_i \in U$, the number of subsets in P (U) that contain x_i is:

$| \{A \subseteq U: x_i \in A\} | = 2^{n-1}$

Thus, the conjecture holds for the power set P (U).

Example: Let $U = \{1, 2, 3, 4, 5, 6\}$

$F = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \ldots, \{1, 2, 3, 4, 5, 6\}\}$

Each element appears in exactly half of the subsets in the power set, regardless of the subset size. In the case above $U = \{1, 2, 3, 4, 5, 6\}$, the power set P (U) contains $2^n = 2^6 = 64$ subsets. Therefore, each element appears in exactly $2^{6-1} = 32$ subsets.

E. F Contains Sets with Symmetry Around an Element

Let F be a union-closed family of subsets of a universal set U, and suppose there exists an element $a \in U$ such that:

$$a \in A \textit{ for } \text{all } A \in F$$

That is, the element a appears in every set of the family. In this case, the total number of sets in which a appears is |F |, which is clearly at least half the size of F. Therefore, a satisfies the condition of Frankl's Union-Closed Sets Conjecture:

$$\exists a \in U \text{ such that } | \{A \in F: a \in A\} | \geq \frac{|F|}{2}$$

Any union-closed family in which a specific element appears in every member trivially satisfies Frankl's Conjecture.

Example:

Let $U = \{1, 2, \ldots, 6\}$ and

$F = \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}, \{1, 4\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 5\}, \{1, 6\}, \{1, 2, 4, 5, 6\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2, 3, 4\}, \{1, 4, 5\}, \{1, 4, 6\}, \{1, 5, 6\}$

Since the element 1 appears in every set of the family, it trivially satisfies Frankl's Conjecture.

F. F Contains Chains (Nested Sets)

Let $U = \{x_1, x_2, \ldots, x_n\}$ be a finite universal set, and let

$F \subseteq$ P (U) be a family of sets such that:

$F = \{A_1, A_2, \ldots, A_k\}$, where $A_1 \subset A_2 \subset \cdots \subset A_k$.

That is, the sets in F form a *chain under inclusion*—a totally ordered collection of sets.

Assume the smallest set in the chain is $A_1 = \{x\}$ for some $x \in U$. Then by construction, each subsequent set in the chain contains x as well. Hence, the element x appears in *every* set in F:

$$x \text{ appears in } |F| \text{ sets} \geq \frac{|F|}{2}.$$

Therefore, Frankl's Conjecture holds trivially for such families.

Example:

Let $U = \{1, 2, \ldots, 6\}$ and

$$\mathcal{F} = \left\{ \{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\}, \{1, 2, 3, 4, 5, 6\} \right\}$$

In this case, the element 1 is contained in every set in the family F, ensuring that the conjecture is satisfied.

G. *F* is Intersection- Closed

Let U be a finite set, and let $F \subseteq P(U)$ be a finite, non- empty family of subsets such that:

i. $A, B \in F \Rightarrow A \cup B \in F$ (union-closed),

ii. $A, B \in F \Rightarrow A \cap B \in F$ (intersection-closed).

Then there exists an element $x \in U$ such that

$$|\{A \in F \mid x \in A\}| \geq \frac{1}{2}|F|.$$

That is, Frankl's Conjecture holds trivially for F.

H. *F* Contains the Empty Set

If the family F consists solely of the empty set, i.e., $F = \{\emptyset\}$, then Frankl's Union-Closed Sets Conjecture holds trivially. Since there are no elements in any set, there are no elements to appear in at least half of the sets. Therefore, the condition of the conjecture is vacuously satisfied.

I. *F* is a Family Built from a Core Set

Let $U = \{x_1, x_2, \ldots, x_n\}$ be a universal set, and let $C \subseteq U$ be a fixed subset referred to as the *core set*. Define a family of sets $F \subseteq P(U)$ such that:

$$\forall A \in F, \quad C \subseteq A$$

That is, every set in F contains the core set C. The family F is union-closed that is for any $A, B \in F$:

$$A \cup B \in F$$

Since all sets in F contain the core set C, each element of C appears in every member of the family. Therefore, every element of C appears in all (and hence at least half of) the sets in F, satisfying Frankl's Conjecture.

Example:

Let $U = \{1, 2, \ldots, 6\}$ and

$$F = \{1, 2, 3\}, \{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 3, 6\}, \{1, 2, 3, 4, 5\}, \{1, 2, 3, 4, 6\}, \{1, 2, 3, 5, 6\}, \{1, 2, 3, 4, 5, 6\}$$

Here, the core set is $C = \{1, 2, 3\}$. Every element in C appears in all sets of the family. Hence, at least one of them must appear in at least half the sets, and the conjecture is trivially satisfied.

The study systematically examined Frankl's Conjecture within union-closed families generated from a finite universal set. Through a detailed analysis of categorized families organized by size, internal structure, and symmetry—the investigation assessed how these attributes influence the likelihood of the conjecture holding. Specifically:

- *Symmetry in Set Distributions:* Families with more evenly distributed set sizes and balanced element occurrences tended to satisfy the conjecture more predictably.
- *Closure Properties:* Certain closure behaviors, such as frequent unions leading to the dominance of specific elements, influenced which element appeared in at least half of the sets.
- *Element Persistence:* Elements that were present in smaller sets early in the formation process often persisted in larger sets, increasing their likelihood of meeting the conjecture's conditions.

4. Conclusion

The study confirmed that specific configurations—such as chains, nested families, and those built around core sub- sets—consistently satisfy the conjecture. In such cases, the appearance of one or more elements in at least half of the sets follows naturally from the structure, rendering the conjecture trivially true.

These findings are significant in that they:

- Help isolate structural patterns that simplify the verification of the conjecture;
- Offer insights into how complexity increases in less structured families;
- Support the broader hypothesis that certain families, due to their construction, inherently satisfy the conjecture.

Generally, the study contributes to a deeper understanding of the structural characteristics that favor or complicate the satisfaction of Frankl's Conjecture.

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