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The Infinitude of Twin Primes

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***Abstract*: The twin prime conjecture inquires whether there are infinitely many prime (p) pairs of the form (p, p+2). This longstanding problem remains unresolved in number theory. Although the infinitude of prime numbers has been rigorously established, the question of infinitely many twin prime pairs has proven considerably more challenging as numerous proposed proofs have been refuted upon closer examination. In this paper, twin primes are classified into three categories according to their final digit. Patterns arising within each category are analyzed, and a proof addressing the infinitude of twin primes is presented.**

***Keywords*: Twin primes.**

1. Introduction

The term twin primes was first introduced by Paul Stäckel in the late nineteenth century to describe pairs of prime numbers that differ by two. The twin prime conjecture, which suggests that there are infinitely many primes p such that $p + 2$ is also prime, remains one of the most profound and challenging open problem in number theory. Despite centuries of dedicated efforts, a complete proof has yet to be found. However, significant progress has been made in understanding the distribution of primes and structures similar to twin primes. A major breakthrough came in 2013 when Zhang [1] proved the existence of infinitely many pairs of distinct primes with gaps not larger than (7×10^7). This result sparked a collaborative effort known as the Polymath Project, which successfully reduced the gap to 246 [2]. Later, Maynard and Tao introduced innovative sieve methods that have significantly advanced analytic number theory [3].

Recent research has concentrated on improving up- per bounds for the twin prime counting function up to a given limit X. Lichtman [4] developed a novel modification of the linear sieve, achieving the best-known upper bounds and surpassing earlier results by Wu (2004). Building on this, Lichtman [5] further improved the level of distribution for primes in arithmetic progressions to moduli as large as $x^{66/107}$, refining estimates related to twin primes and Goldbach-type problems. Pascadi [6] extended these advances by employing triply-well-factorable weights, yielding new distribution exponents and sharper bounds. Meanwhile, Li [7] strengthened Chen's theorem by improving representations of even numbers as the sum of a prime and an almost prime, indirectly contributing to the understanding of twin primes.

Alongside these rigorous analytic approaches, heuristic and probabilistic models continue to provide valuable insights. For example, Bufalo and Iavernaro [8] introduced probabilistic frameworks to estimate the likelihood of twin primes within specified intervals, providing further support for the conjecture's plausibility. While these models do not constitute proofs, they complement analytic progress by offering intuitive perspectives.

Despite these significant advances, the question of the infinitude of twin primes remains open. Nevertheless, ongoing refinements in sieve techniques, distribution results, and computational models keep the momentum alive in this vibrant area of modern number theory, bringing the mathematician ever closer to resolving this longstanding conjecture. This research paper categorizes twin primes into three subclass categories based on the last digit. This categorization helped in identifying patterns within each group, which in turn assisted in proving the conjecture.

2. Preliminaries

Definition 2.1 (Prime number):

An integer $p > 1$ is a **prime number** if and only if its only divisors are ± 1 and $\pm p$.

Example 2.0.1:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29,.. are examples of prime numbers.

Definition 2.2 (Twin prime):

Two primes p, q forms a twin prime pair if and only if $q = p + 2$.

Example 2.0.2:

(11, 13), (17, 19), (29, 31),.. are examples of twin primes.

Proposition 2.1:

All primes greater than three are of the form $6K \pm 1$, where K is a positive integer.

Proof 2.2:

Let r be a prime number greater than three. Consider the two integers $r - 1$ and $r + 1$. Since r is prime number, then $r - 1$ and $r + 1$ are even. Among any three-consecutive number, one must be divisible by 3; therefore, either $r - 1$ or $r + 1$ is divisible by 3. Since both $r - 1$ and $r + 1$ are even (divisible by 2) and one of the two also is divisible by 3, it follows that one of this numbers is divisible by 6. Hence

i. If $r - 1$ is divisible by 3, then $r - 1$ is in the form of $6K$, where $6 = 2 \cdot 3$, and therefore $r = 6K + 1$.
ii. If $r + 1$ is divisible by 3, then $r + 1$ is also in the form of $6K$, where $6 = 2 \cdot 3$, and therefore $r = 6K - 1$.

Therefore, $r = 6K \pm 1$, which completes the proof.

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Proposition 2.3:

All twin primes except (3, 5) are of the form ($6K - 1$, $6K + 1$), where K is a positive integer.

Proof 2.4:

Let (W, Z) be any twin prime pair except (3, 5), and let X be the middle number between W and Z. For example, for the pair (11, 13), the middle number X is 12. Among every three consecutive numbers (W, X, Z) one is divisible by three. Since, W and Z are prime numbers, neither can be divisible by 3, so X must be divisible by three. Also, one integer between two consecutive numbers is even. Therefore, either W or X is even. Since, W is prime, it cannot be even, so, X must be even. This implies that X has 2 and 3as its factors. So, $X = (2 \cdot 3 \cdot K) = 6K$ where K is a positive integer. Thus, with respect to X, we have

$W = 6K - 1$, $Z = 6K + 1$

And so, the pair (W, Z) is of the form ($6K - 1$, $6K + 1$).

3. Twin Primes Categorization

Notation: (a, b) denotes a twin prime pair whose first coordinate ends with a, and second coordinate ends with b.

All prime numbers p are odd numbers (except 2), and their only divisors are ± 1 and $\pm p$. This implies that any integer ending with 0, 2, 4, 6, and 8 cannot be

a prime number since it is divisible by 2. Also, any number ending with 5 is divisible by 5 and therefore, excluding 5 itself (which is prime), any other number ending with 5 cannot be a prime number. Hence, it is clear that all primes apart from 2 and 5 ends with; 1, 3, 7 or 9.

Example 3.0.1: 1<u>1</u>, 1<u>3</u>, 1<u>7</u>, 1<u>9</u>, 2<u>3</u>, 2<u>9</u>, 3<u>1</u>, 3<u>7</u>, ...

Since twin primes are primes, any twin prime pairs apart from (3, 5) and (5, 7) ends with 1, 3, 7 or 9. Due to this, twin primes have been classified into three categories based on these last digits (call it, Categories of Twin Primes):

i. First category: This category consists of those twin primes of the form (1, 3).

Example 3.0.2: (1<u>1</u>, 1<u>3</u>), (4<u>1</u>, 4<u>3</u>), (7<u>1</u>, 7<u>3</u>) and (10<u>1</u>, 10<u>3</u>) (call it, Cat1).

ii. Second category: This category consists of twin primes of the form (7, 9).

Example 3.0.3: (1<u>7</u>, 1<u>9</u>), (10<u>7</u>, 10<u>9</u>), (19<u>7</u>, 19<u>9</u>) and (22<u>7</u>, 22<u>9</u>). (call it, Cat2).

iii. Third category: This consists of those twin primes of the form (9, 1).

Example 3.0.4: (2<u>9</u>, 3<u>1</u>), (5<u>9</u>, 6<u>1</u>), (14<u>9</u>, 15<u>1</u>) and (17<u>9</u>, 18<u>1</u>) (call it, Cat3).

We therefore have three categories of twin primes; the first category (Cat1), second category (Cat2) and the third category (Cat3).

Example 3.0.5:

Table 1
Categories of twin primes

Cat1	Cat2	Cat3
(11, 13)	(17, 19)	(29, 31)
(41, 43)	(107, 109)	(59, 61)
(71, 73)	(137, 139)	(149, 151)
(101, 103)	(197, 199)	(179, 181)
(191, 193)	(227, 229)	(239, 241)
(281, 283)	(347, 349)	(269, 271)
(311, 313)	(617, 619)	(419, 421)
(431, 433)	(827, 829)	(569, 571)
(461, 463)	(857, 859)	(599, 601)
(521, 523)	(1277, 1279)	(659, 661)

Note:

i. If p is a prime, where $p \neq 2\&5$, then p ends with digit 1, 3, 7 or 9.

ii. (11, 13), (17, 19) and (29, 31) are the first in cat1, cat2 and cat3 respectively, we therefore set them as the base for each category.

Claim 3.1:

Let {a, b, c, ...} and {a_1, b_1, c_1, ...} be two different sets of consecutive integers then, $a_1 - a = b_1 - b = c_1 - c = ...$

Proof 3.2:

Let t, t+1, t+2 be three consecutive integers and r, r+1, r+2 be other three consecutive integers. Then

$$(t - r) = ((t+1) - (r+1)) = ((t+2) - (r+2)) = (t-r)$$

Example 3.2.1:

Given (55, 56, 57) and (22, 23, 24) find the difference consecutively

55 - 22 = 33

56 - 23 = 33

57 - 24 = 33

Thus (55 - 22) = (56 - 23) = (57 - 24) = 33

Lemma 3.3: (Kiloni)

For each of the three categories of twin primes, there exist a positive integer n such that the difference between any two twin prime pairs is $30n$.

Proof 3.4:

From proposition 2.3, let ($6K - 1$, $6K + 1$) and ($6Z - 1$, $6Z + 1$) be any two pairs of twin primes from the same category, where K and Z are positive integers and ($6Z - 1$, $6Z + 1$) > ($6K - 1$, $6K + 1$). Then from claim 3.1, ($6Z - 1$) - ($6K - 1$) = ($6Z + 1$) - ($6K + 1$) which is equal to $6Z - 6K$. Let T = $6Z - 6K = 6(Z - K)$, then T = $6M$, where $M = Z - K$. But, ($6K - 1$, $6K + 1$) and ($6Z - 1$, $6Z + 1$) are from the same category and ends with the same digits, hence finding the difference gives a result which end with a zero digit (Example: 26 - 16 = 10) and thus, implies that $6M$ end with a zero digit. Thus, $6M$ can either be 30, 60, 90, ... Therefore, M is multiples of 5. Which is the same

as: $6 \cdot (5n) = 30n$, where n is any positive integer. Implying that the difference between ($6K - 1$, $6K + 1$) and ($6Z - 1$, $6Z + 1$), (which are any two twin prime pairs from the same category) is given by 30n.

NOTE: This proof applies for all the categories. From the sample on TABLE 1, one can observe that, the difference between any two pairs is a multiple of 30. As described below on example 3.4.1, 3.4.2, and 3.4.3.

Example 3.4.1:

A. Cat1

let $(6K - 1, 6K + 1) = (11, 13)$ and $(6Z - 1, 6Z + 1) = (101, 103)$

Then $(6Z + 1) - (6K + 1) = (6Z - 1) - (6K - 1)$

⇒ 101 - 11 = 103 - 13 = 90

⇒ T = 90 = 6(M=15) = 6(5 · (n= 3) = 30(3)

Therefore, the difference between (11, 13) and (101, 103) is equal to 90, which is in the form of 30n, where n = 3.

Example 3.4.2:

B. Cat2

let $(6K - 1, 6K + 1) = (17, 19)$ and $(6Z - 1, 6Z + 1) = (197, 199)$

Then $(6Z + 1) - (6K + 1) = (6Z - 1) - (6K - 1)$

⇒ 197 - 17 = 199 - 19 = 180

⇒ T = 180 = 6(M=30) = 6(5· (n= 6) = 30(6).

Therefore, the difference between (17, 19) and (197, 199) is equal to 180, which is in the form of 30n, where n=6.

Example 3.4.3:

C. Cat3

let $(6K - 1, 6K + 1) = (9239, 9241)$ and $(6Z - 1, 6Z + 1) = (9929, 9931)$

Then $(6Z + 1) - (6K + 1) = (6Z - 1) - (6K - 1)$

⇒ 9929 - 9239 = 9931 - 9241 = 690

⇒ T = 690 = 6(M=115) = 6(5· (n= 23) =30(23)

Therefore, the difference between (9239, 9241) and (9929, 9931) is equal to 690, which is in the form of 30n, where n=23.

Theorem 3.5:

Twin primes are infinitely many.

Proof 3.6:

Assume that (p, p+2) is the largest existing twin prime pair. From lemma 3.3, there exists an integer $n > 0$ such that the difference between any two twin primes from the same category is given by 30n. This implies that there must exists an integer $n > 0$ such that when $30n$ is added to $(p, p + 2)$ gives another twin prime pair of the form $(p', p' + 2)$ $(\exists\ n \in Z^+ \mid 30n + (p, p + 2) = (p', (p' + 2))$, where $(p', (p' + 2))$ is another pair of twin prime generated). Since $(p', p' + 2) = (p, p + 2) + 30n \Longrightarrow (p', p' + 2) > (p, p + 2)$. This contradicting the first statement that (p, p+2) is the largest twin prime pair. Therefore, twin primes are infinite.

Example 3.6.1:

1) Cat1

Assume (71, 73) is the largest existing twin prime. Then:

Solution:

There must exists n such that; $(71, 73) + 30n = (p', (p' + 2))$

Starting with n = 1; then (71, 73) + 30 = (101, 103)

Both 101 and 103 are primes, thus (101, 103) twin prime pair (101, 103) > (71, 73) hence, (71, 73) is not the largest existing twin prime.

2) Cat2

Assume (3000377, 3000379) is the largest existing twin prime pair. Then:

Solution:

There must exists n such that; $(3000377, 3000379) + 30n = (p', (p' + 2))$

Taking n = 1

Then (3000377, 3000379) + 30 = (3000407, 3000409)

3000407 is composite while 3000409 is prime, thus, (3000407, 3000409) is not a twin prime pair.

Let n = 2

(3000377, 3000379) + 60 = (3000437, 3000439)

Both 3000437 and 3000439 are composite, cannot be a twin prime pair, thus the process continues;

....

....

....

Taking n = 51 Then;

(3000377, 3000379) + 1530 = (3001907, 3001909)

Both are primes, hence, twin prime pair.

(3001907, 3001909) > (3000377, 3000379).

Thus (3000377, 3000379) is not the largest existing twin prime.

3) Cat3

Assume (100000469, 100000471) is the largest existing twin prime. Then:

Solution:

There must exists n such that: $(100000469, 100000471) + 30n = (p', (p' + 2))$

Taking n = 1

Then (100000469, 100000471) + 30 = (100000499, 100000501)

Both 100000499 and 1000004101 are composite, thus, (100000499, 1000004101) is not a twin prime pair.

Let n = 2

(100000469, 100000471) + 60 = (100000529, 100000531)

100000529 is composite while 100000531 is prime, thus the process continues;

....

....

.... n = 11

Then,

(100000469, 100000471) + 330 = (100000799, 100000801)

Both are primes, hence, twin prime pair.

(100000469, 100000471) < (100000799, 100000801).

Thus, (100000469, 100000471) is not the largest existing twin prime.

4. Conclusion

All prime numbers except 2 and 5 ends with 1, 3, 7 or 9. As infinity is approached, prime numbers tend to thin out and so do the twin prime pairs. This does not imply that, as twin primes become rarer as we approach infinity, they are finite. Based on the last digits it is clear that twin primes have been categorized into three categories; Cat1, Cat2 and Cat3. From the proof of Theorem 3.5, it is evident that it is impossible to arrive at the largest twin prime, since there exists n such that when $30n$ is added to any twin prime pair gives another larger pair. The proof also guarantees the immediate next twin prime pair along the categories. It is also clear that, there exists so many n from 1 to infinity that satisfy the equation $(p, p{+}2) + 30n = (p', p' + 2)$ (i.e. starting from any base of each category, it is possible to generate other pairs of twin primes along the base category by changing the value of n). It is possible to find larger twin primes no matter how large (p, p+2) is. This is a proof by contradiction of the infinitude of twin primes, for our assumption of (p, p+2) as the largest existing twin prime pair will be contradicted by the discovery of largest twin prime pair, implying that there are infinitely many twin primes.

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