New Forms of Conformable Fractional Derivative Using Backward and Central Difference

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***Abstract*: In this paper we define two new forms of conformable fractional derivative formed by using backward and central difference. These forms satisfy Linearity property, product rule, Quotient rule, and derivative constant is zero. And these forms satisfy all the properties by conformable fractional derivative. (using forward difference). These forms satisfy the previous results for ordinary derivative and derivative of some standard function. This define forms coinside with the classical definition of first order derivative.**

***Keywords*: backward difference, central difference, conformable fractional derivative, fractional derivative new forms of conformable fractional derivative.**

1. Introduction

Fractional derivative is as old as calculus. L'Hospital in 1695 asked to Libniz what does it mean if $\frac{d^n f}{dx^n}$.if n is fraction. Since then, many researchers tried to put a definition of a fractional derivative, but many definition Satisfies linearity property but fails to satisfies other properties of first order derivative. Most of them used an *integral form* for the fractional derivative.

2. Methodology

Definition 1) (forward difference definition): Let *f:* $[0, \infty) \to$ R and $t > 0$. Then the definition of the derivative of f at t is $\frac{df}{dt} = \lim_{\epsilon\to 0}\frac{f\,(t+\varepsilon)-f\,(t)}{\epsilon}$.

This is the definition using forward difference.

Definition 2) backward difference definition: Let *f:* $[0, \infty) \to$ R and $t > 0$. Then the definition of the derivative of f at t is $\frac{df}{dt} = \lim_{\epsilon\to 0}\frac{f\,(t)-f\,(t-e)}{\epsilon}$.

This is the definition using backward difference.

Definition 3) central difference definition: Let $f : [0, \infty) \to$ R and $t > 0$. Then the definition of the derivative of f at t is $\frac{df}{dt} = \lim_{\epsilon\to 0}\frac{f\,(t+\varepsilon)-f\,(t-\epsilon)}{2\epsilon}$.

This is the definition using central difference.

All the three forms are equal.

Let us write $T_1 = \frac{df}{dt}$ derivative of f.

3. Models and Analysis

All the three forms are equal and satisfies following properties:

(i) $T_1(af + bg) = aT_1(g) + bT_1(f)$, for all $a, b \in$ R and f, g in the domain of T_1.

(ii) $T_1(t^p) = pt^{p-1}$

(iii)$T_1(fg) = fT_1(g) + gT_1(f)$

(iv)$T_1(f/g) = \frac{gT_1(f)-fT_1(g)}{g^2}$

(v)$T_1(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

Now, new definition of conformable fractional derivatives given for order $\alpha \in (0, 1]$. [5] And generalized definition to any $\alpha \in (n, n+1]$. [5]

Definition 1. Conformable fractional derivative using forward difference for $\alpha \in (0,1]$:*[5]* Given a function $f : [0, \infty) \dashrightarrow$ R. Then the ''*conformable fractional derivative*'' of f of order $\alpha \in (0,1]$ is defined by

$$T_\alpha f(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}, \text{ [5]}$$

This definition satisfy all the above five properties of T_1 of derivative.

For $\alpha = 1$

$$T_1\mathrm{f(t)} = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-1})-f(t)}{\varepsilon}$$

$$= \lim_{\epsilon\to 0}\frac{f\,(t+\varepsilon)-f\,(t)}{\epsilon}$$

Reduces to classical definition of derivative.

Definition 1. Conformable fractional derivative using forward difference for $\alpha \in (n, n+1]$ *[5]:* Let if $\alpha \in (n, n+1]$, and f be an n-differentible at t,where $t > 0$ then the new definition of comformable fractional derivative of f of order α is defined as,

$$T_\alpha f(t) = \lim_{\varepsilon\to 0}\frac{f^{(\lceil\alpha\rceil-1)}(t+\varepsilon^{\lceil\alpha\rceil}t^{(\lceil\alpha\rceil-\alpha)}-f^{(\lceil\alpha\rceil-1)}(t)}{\varepsilon^{\lceil\alpha\rceil}} \quad [5]$$

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Where $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α.

Remark 1.1: One can easily prove that,

$$T_\alpha f(t) = t^{(\lceil\alpha\rceil-\alpha)} f^{\lceil\alpha\rceil}(t)$$

Where $\alpha \in (n, n+1]$ and f is n+1 differentiable at $t > 0$

4. Results and Discussion

New forms definition of Conformable fractional derivative using backward and central difference.

Definition 2(backword difference definition for $\alpha \in (0,1]$*).* Given a function *f:* [0, ∞*)* -→ R. Then the ‘‘*conformable fractional derivative*’’ by using backword difference of *f* of order *α* ∈ (0,1] is defined by,

$$T_\alpha f(t) = \lim_{\varepsilon\to 0} \frac{f(t) - f(t - \varepsilon t^{1-\alpha})}{\varepsilon}$$

This definition satisfy all the above five properties of T_1 of derivative.

For $\alpha = 1$

$$T_1 \mathrm{f(t)} = \lim_{\varepsilon\to 0} \frac{f(t) - f(t - \varepsilon t^{1-1})}{\varepsilon}$$
$$= \lim_{\epsilon\to 0} \frac{f\ (t) - f\ (t - \varepsilon)}{\epsilon}$$

Reduces to classical definition of derivative.

For all *t* > 0, *α* ∈ *(*0, 1*)*. If *f* is *α*-differentiable in some *(*0, *a)*, *a* > 0, and $\lim_{t\to 0^+} f^\alpha(t)$ exists then define $\lim_{t\to 0^+} f^\alpha(t) = f^\alpha(0)$.

$f^\alpha(t)$ = $T_\alpha f(t)$ write to denote the conformable fractional derivatives of *f* of order *α*.

In addition, if the conformable fractional derivative of *f* of order *α* exists, then we simply say *f* is *α*-differentiable.

These definition is for range $\alpha\ is\ (0,1)$ now $\alpha \in (n, n+1]$ for some natural number n the generalized definition is .

Definition 2 (backword difference definition for $\alpha \in (n, n+1]$*):* Let if $\alpha \in (n, n+1]$, and f be an n-differentible at t,where $t > 0$ then the new definition of comformable fractional derivative using backword difference of f of order α is defined as,

$$T_\alpha f(t) = \lim_{\varepsilon\to 0} \frac{f^{(\lceil\alpha\rceil-1)}(t) - f^{(\lceil\alpha\rceil-1)}(t - \varepsilon t^{(\lceil\alpha\rceil-\alpha)})}{\varepsilon^{\lceil\alpha\rceil}}$$

Where $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α.

Remark 2.1: one can easily prove that,

$$T_\alpha f(t) = t^{(\lceil\alpha\rceil-\alpha)} f^{\lceil\alpha\rceil}(t)$$

Where $\alpha \in (n, n+1]$ and f is n+1 differentiable at $t > 0$

Definition 3. (Central difference definition for $\alpha \in (0,1]$: Given a function f : [0, ∞*)* -→ R. Then the “*conformable fractional derivative*” by using central difference of *f* of order *α* ∈ (0,1] is defined by,

$$T_\alpha f(t) = \lim_{\varepsilon\to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t - \varepsilon t^{1-\alpha})}{2\varepsilon}$$

This definition satisfy all the above five properties of T_1 of derivative.

For $\alpha = 1$

$$T_1 \mathrm{f(t)} = \lim_{\varepsilon\to 0} \frac{f(t + \varepsilon t^{1-1}) - f(t - \varepsilon t^{1-1})}{\varepsilon}$$
$$= \lim_{\epsilon\to 0} \frac{f\ (t + \varepsilon) - f\ (t - \varepsilon)}{2\epsilon}$$

Reduces to classical definition of derivative.

For all *t* > 0, *α* ∈ *(*0, 1*)*. If *f* is *α*-differentiable in some *(*0, *a)*, *a* > 0, and $\lim_{t\to 0^+} f^\alpha(t)$ exists then define $\lim_{t\to 0^+} f^\alpha(t) = f^\alpha(0)$.

$f^\alpha(t)$ = $T_\alpha f(t)$ write to denote the conformable fractional derivatives of *f* of order *α*.

In addition, if the conformable fractional derivative of *f* of order *α* exists, then we simply say *f* is *α*-differentiable.

These definition is for range $\alpha\ is\ (0,1)$ now $\alpha \in (n, n+1]$ for some natural number n the generalized definition is .

Definition 3 central difference definition for $\alpha \in (n, n+1]$: Let if $\alpha \in (n, n+1]$, and f be an n-differentible at t,where $t > 0$ then the new definition of comformable fractional derivative using backword difference of f of order α is defined as,

$$T_\alpha f(t) = \lim_{\varepsilon\to 0} \frac{f^{(\lceil\alpha\rceil-1)}(t + \varepsilon t^{(\lceil\alpha\rceil-\alpha)}) - f^{(\lceil\alpha\rceil-1)}(t - \varepsilon t^{(\lceil\alpha\rceil-\alpha)})}{\varepsilon^{\lceil\alpha\rceil}}$$

Where $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α.

Remark 3.1: one can easily prove that,

$$T_\alpha f(t) = t^{(\lceil\alpha\rceil-\alpha)} f^{\lceil\alpha\rceil}(t)$$

Where $\alpha \in (n, n+1]$ and f is n+1 differentiable at $t > 0$

5. Conclusion

The first form of definition conformable fractional derivative is given by using forward difference [5]. The second form of definition conformable fractional derivative given by backward difference. The third form of conformable fractional derivative definition is given by central difference.

All the three forms of conformable fractional derivative are equivalent and satisfies the properties and theorem of derivative given in [5].

References

[1] K. S. Miller, An Introduction to Fractional Calculus and Fractional Differential Equations, J. Wiley and Sons, New York, 1993.

[2] K. Oldham, J. Spanier, The Fractional Calculus, Theory and Applications of Differentiation and Integration of Arbitrary Order, Academic Press, USA, 1974.

[3] A. Kilbas, H. Srivastava, J. Trujillo, Theory and Applications of Fractional Differential Equations, in: Math. Studies., North-Holland, New York, 2006.
[4] I. Podlubny, Fractional Differential Equations, Academic Press, USA, 1999.
[5] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh "A new Definition of Fractional derivative", journal of computational and applied mathematics.